Journal of Physics: Conference Series

PAPER • **OPEN ACCESS**

Spectrum properties of self-adjoint operator

To cite this article: A Ekayanti and C R Indrati 2020 *J. Phys.: Conf. Ser.* **1517** 012053

Spectrum properties of self-adjoint operator

A Ekayanti[1*] and C R Indrati[2]

[1] Universitas Muhammadiyah Ponorogo, Ponorogo, Indonesia
[2] Universitas Gadjah Mada, Yogyakarta, Indonesia

*Email: arta_ekayanti@ymail.com

Abstract. In this paper, we discuss about spectrum properties of self-adjoint operator. Relation between self-adjoint operator and normal operator is used to show that the spectrum of self-adjoint operator is subset of ℝ. Furthermore, by using norm properties of bounded-linear operator and real polynomial we derived the relation between spectrum and norm of operator polynomial.

1. Introduction

Functional analysis is a branch of mathematics. The concept of eigenvalues of matrix play fundamental role in functional analysis. Linear operators in complex vector spaces always have a representation matrix, so that the spectral theory in finite dimension vector space equivalent with the spectral theory of the matrix. In other side, linear operators in infinite dimension complex vector spaces do not have a representation matrix. The notion of the eigenvalue was generalized from finite dimension of vector space to infinite dimension. It means, the set of Eigen values of a matrix was generalized to be spectrum of a linear continuous operator.

This shows how important spectrum is. This is supported by Mendelson and Tomberg which states that some discussion about spectrum will be necessary later in the development of functional analysis studies [1].

Definition 1.1 [2]
If V and W are vector spaces, a mapping $A: V \to W$ is said to be linear in case
i. A is additive : $A(x+y) = Ax + Ay$, for all $x, y \in V$.
ii. A is homogeneous: $A(\lambda x) = \lambda(Ax)$ for all $x \in V$ and scalar λ.

Definition 1.2.
If V and W are normed spaces, a linear mapping $A: V \to W$ is said to be bounded if there is $c \in \mathbb{R}$ such that $\|Ax\| \leq c\|x\|$ for all $x \in V$.

There is a theorem about the relation between linear continuous mapping and bounded linear mapping. The theorem as following:

Theorem 1.1. [2]
Let V and W be normed spaces, $A: V \to W$ a linear mapping. The following condition on A are equivalent:
i. A is a continuous mapping.
ii. A is continuous at some point $x_0 \in V$.
iii. A is continuous at $\theta \in V$.
iv. $\{\|Ax\|: \|x\| \leq 1\}$ is bounded set of real numbers.

v. There exists a constant $M \geq 0$ such that $\|Ax\| \leq M\|x\|$ for all $x \in V$.

In view of this theorem, continuous linear mappings between normed spaces are also bounded linear mappings. We know that a mapping between normed spaces is called operator. Furthermore, in this paper bounded linear mapping or continuous linear mapping are called bounded operator.

Let H be Hilbert space. A collection of bounded operator $A: H \to H$ denote by $B(H)$. Note that, for every bounded operator $A: H \to H$ there is $A^*: H \to H$ operator such that for every $x, y \in H$ we have$\langle A(x), y\rangle = \langle x, A^*(y)\rangle$. Furthermore, A^* is called adjoint operator.

Definition 1.3. [3]
An operator $A \in B(H)$ is said to be normal in case $A^*A = AA^*$/

Definition 1.4. [3]
Operator $A \in B(H)$ such that $A = A^*$ is called self-adjoint operator.

2. Discussion

Definition 2.1 [4]
Suppose $A \in B(H)$. The spectrum of A denoted by $\sigma(A)$ is defined by

$$\sigma(A) = \{\lambda \in \mathbb{C}: A - \lambda I \ not\ invertible\} \tag{1}$$

Furthermore, complex numbers λ is called *proper values* for operator A if there is $x \neq \theta$ such that $Ax = \lambda x$. Equivalently, $\lambda \in \mathbb{C}$ is proper value for operator A if and only if there is $x \neq \theta$ such that $\|Ax - \lambda x\| = 0$. From this properties, we can define *approximate proper value* or AP-value for operator A as a collection of λ such that for every $\varepsilon > 0$ there exists $x \in H, \|x\| = 1$ such that $\|Ax - \lambda x\| < \varepsilon$. Suppose that $x \in H, \|x\| \neq 0$ we can define

$$x' = \frac{x}{\|x\|} \tag{2}$$

thus we have $\|x'\| = 1$, therefore

$$\|Ax' - \lambda x'\| < \varepsilon \tag{3}$$

$$\left\|A\frac{x}{\|x\|} - \lambda\frac{x}{\|x\|}\right\| < \varepsilon \tag{4}$$

$$\|Ax - \lambda x\| < \varepsilon\|x\| \tag{5}$$

Furthermore, we can defined $\Pi(A) = \{\lambda \in \mathbb{C}: \lambda \text{ approximate proper value for operator } A\}$ is said to be *approximate point spectrum* for operator A or AP-spectrum.-

Theorem 2.1
If $A \in B(H)$, then $\Pi(A) \subseteq \sigma(A)$.
Proof. Suppose that $\lambda \notin \sigma(A)$, thus $A - \lambda I$ invertible. Consequently, for every $x \in H$ we have

$$\|x\| = \|(A - \lambda I)^{-1}(A - \lambda I)x\| \leq \|(A - \lambda I)^{-1}\|\|(A - \lambda I)x\|. \tag{6}$$

Therefore, for every $x \in H$ we have $\|Ax - \lambda x\| \geq \varepsilon\|x\|$ for $\varepsilon = \frac{1}{\|(A-\lambda I)^{-1}\|}$. Briefly, $\lambda \notin \Pi(A)$.

Theorem 2.1 show that $\Pi(A) \subseteq \sigma(A)$ for every $A \in B(H)$. It will be valid for the contrary if A is normal operator. It can be seen on theorem 2.2 as below:

Theorem 2.2.
If $A \in B(H)$ normal operator, then $\Pi(A) = \sigma(A)$.
Proof. It follows from Theorem 2.1 that $\Pi(A) \subseteq \sigma(A)$. So, we need only prove $\sigma(A) \subseteq \Pi(A)$. Let $\lambda \notin \Pi(A)$, then there exists $\varepsilon > 0$ such that for every $y \in H$ we have

$$\|Ay - \lambda y\| \geq \varepsilon\|y\|. \tag{7}$$

Since A is normal operator, it follows that $A - \lambda I$ be normal operator. Consequently, for every $y \in H$, we have

$$\|A^*y - \lambda y\| \geq \varepsilon\|y\|. \tag{8}$$

Suppose that y be orthogonal vector of the range of $A-\lambda I$, therefore for every $x \in H$ it follows $\langle (A-\lambda I)x, y\rangle = 0$. Furthermore, we have $\langle (A^* - \bar{\lambda})x, y\rangle = 0$. Consequently, since

$$0 = \|A^* y - \lambda y\| \geq \varepsilon \|y\| \tag{9}$$

it follows that $y = 0$. On other words, $(A-\lambda I)$ is invertible and therefore $\lambda \notin \sigma(A)$.

Beside above properties, there is some other properties of bounded operator. We can look on the following theorem:

Theorem 2.3.
If $A \in B(H)$ such that $\|I - A\| < 1$, then A is invertible.
Proof. Define $\alpha = 1 - \|I - A\|$, it follows $\alpha > 0$. Suppose $x \in H$, then

$$\|A(x)\| = \|x - (x - A(x))\| \geq \|x\| - \|x - A(x)\| = (1 - \|I - A\|)\|x\| = \alpha\|x\|. \tag{10}$$

Therefore, A is bounded below. Define $M = \{y \in H | y = A(x), x \in H\}$ and $\delta = \inf\{\|x - y\| | x \in H, y \in M\}$. We will show that $\delta = 0$. Suppose otherwise, $\delta > 0$. Since $1 - \alpha < 1$, there exists $x \in H$ and $y \in M$ such that $\|x - y\| < \frac{\delta}{1-\alpha}$. It follows that

$$\delta \leq \|(x-y) - A(x-y)\| \leq \|I - A\|\|x - y\| = (1 - \alpha\|x - y\|) < \delta. \tag{11}$$

It means, $\delta < \delta$ which is a contradiction. Therefore, we can conclude that $\delta = 0$.

Theorem 2.4.
Let $A \in B(H)$, $\lambda \in \sigma(A)$ then $|\lambda| \leq \|A\|$.
Proof. Suppose that $\lambda > \|A\|$. It was seen from Theorem 2.3, it follows $I - \frac{A}{\lambda}$ is invertible. Consequently, $A - \lambda$ is invertible. That is $\lambda \notin \sigma(A)$.

Theorem 2.5.
Let $A \in B(H)$ self-adjoint operator, then $\sigma(A) \subseteq \mathbb{R}$.
Proof. Suppose $\lambda \notin \mathbb{R}$, thus for every $x \neq 0$ we have

$$|\lambda - \bar{\lambda}|\|x\|^2 > 0 \Leftrightarrow |\bar{\lambda} - \lambda\,|\langle x, x\rangle > 0 \Leftrightarrow \langle (|\bar{\lambda} - \lambda\,|)x, x\rangle > 0 \Leftrightarrow |\langle \bar{\lambda}x, x\rangle - \langle \lambda x, x\rangle| > 0 \tag{12}$$

Moreover, we have $|\langle (A-\lambda)x, x\rangle - \langle (A-\bar{\lambda})x, x\rangle| > 0$. Consequently,

$$2\|Ax - \lambda x\|\|x\| > 0 \tag{13}$$

That is $\|Ax - \lambda x\| > 0$, for every $x \in H$. Since A self-adjoint operator, then A is normal operator. Moreover, by Theorem 2.2 we have $x \notin \sigma(A)$.

Theorem 2.6.
Let $A \in B(H)$ and p is polynomial, then

$$\sigma(p(A)) = p(\sigma(A)) = \{p(\lambda) | \lambda \in \sigma(A)\}. \tag{14}$$

Proof. Suppose that $\lambda_0 \in \mathbb{C}$. We know that $\lambda = \lambda_0$ is roots of $p(\lambda) = p(\lambda_0)$, moreover there exists q polynomial such that $p(\lambda) - p(\lambda_0) = (\lambda - \lambda_0)q(\lambda)$. Similarly, we have

$$p(A) - p(\lambda_0) = (A - \lambda_0)q(\lambda). \tag{15}$$

Since $\lambda_0 \in \sigma(A)$, it follows $A - \lambda_0$ is not invertible. Therefore, $p(A) - p(\lambda_0)$ is not invertible, that is $p(\lambda_0) \in \sigma(p(A))$. Consequently, $p(\sigma(A)) \subseteq \sigma(p(A))$. Moreover, suppose $\lambda_0 \in \sigma(p(A))$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ roots of $p(\lambda) = \lambda_0$. Therefore, we have

$$p(A) - \lambda_0 = \alpha(A - \lambda_1)(A - \lambda_2) \ldots (A - \lambda_n), \tag{16}$$

for scalar $\alpha \neq 0$. Since $\lambda_0 \in \sigma(p(A))$, then there is at least one j with $1 \leq j \leq n$ such that $A - \lambda_j$ is not invertible. For this j, $\lambda_j \in \sigma(A)$ and $p(\lambda_j) = \lambda_0$, therefore $p(\lambda_j) \in \sigma(p(A))$. Consequently, $p(\sigma(A)) \subseteq \sigma(p(A))$. So, we can conclude that $p(\sigma(A)) = \sigma(p(A)) = \{p(\lambda) | \lambda \in \sigma(A)\}$.

As theorem 2.5, the following theorem give the properties of spectrum of self-adjoint operator.

Theorem 2.7.
Let $A \in B(H)$ self-adjoint operator then $\|A\| = \sup\{|\lambda| : \lambda \in \sigma(A)\}$.

Proof. Define $\alpha \coloneqq \sup\{|\lambda|: \lambda \in \sigma(A)\}$. By Theorem 2.4 we have $\alpha \leq \|A\|$. Furthermore, we have to prove that $\|A\| \leq \alpha$. It can prove by using $\|A\|^2 \in \Pi(A^2)$. For every $x \in H$ we have

$$\|A^2(x) - \lambda^2 x\|^2 = \|A^2(x)\| - 2\lambda^2\|A(x)\|^2 + \lambda^4\|x\|^2, \quad (17)$$

for $\lambda \in \mathbb{R}$. Let $(x_n) \subseteq H$ be sequence of unity vector such that $\|A(x_n)\| \to \|A\|$ for every $n \to \infty$ and $\lambda = \|A\|$, then for $n \to \infty$ we have

$$\|A^2(x_n) - \lambda^2 x_n\| \leq (\|A\|\|A(x_n)\|)^2 - 2\lambda^2\|A(x_n)\|^2 + \lambda^4 = \lambda^4 - \lambda^2\|A(x_n)\|^2 \to 0. \quad (18)$$

Therefore, we have conclussion that $\|A\|^2 \in \Pi(A^2)$.

From theorem 2.6 and theorem 2.7, we can derived the relation between spectrum and norm of operator polynomial as the following theorem.

Theorem 2.8.

Let $A \in B(H)$ self-adjoint operator and p be a real polynomial, then

$$\|p(A)\| = \sup\{|p(A)|: \lambda \in \sigma(A)\}. \quad (19)$$

Proof. By using Theorem 2.7, for $A \in B(H)$ self-adjoint operator we have $\|A\| = \sup\{|\lambda|: \lambda \in \sigma(A)\}$. Therefore, we get $\|p(A)\| = \sup\{|\lambda|: \lambda \in \sigma\big(p(A)\big)\}$. Based on Theorem 2.6,

$$\sigma\big(p(A)\big) = p\big(\sigma(A)\big) = \{p(\lambda): \lambda \in \sigma(A)\}. \quad (20)$$

Furthermore, we obtained

$$\|p(A)\| = \sup\{|\lambda|: \lambda \in p\big(\sigma(A)\big)\} = \sup\{|p(A)|: \lambda \in \sigma(A)\}. \quad (21)$$

Conclusion

The spectrum of self-adjoint operator is subset of $\mathbb{R}$. Self-adjoint operator has relation with real polynomial, for $A \in B(H)$ self-adjoint operator and p be a real polynomial, we have

$$\|p(A)\| = \sup\{|p(A)|: \lambda \in \sigma(A)\}.$$

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